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B.Sc.-part-I (H), paper-I
Set Theory Date- 21-04-2020.

Q.1 Define partial order, linear order and well order on a set with an example of each one of them.

Ans - Partial Order: $\rightarrow$ Let X is any non-empty set and R is a relation in X , then R is called a partial order relation in X when R is antisymmetric and transitive. If we write R as $\leq$, then ' $\leq$ ' is a partial order if:

(I) For every $x, y \in X$, if $x \leq y$ and $y \leq x$ then $x = y$
[anti-symmetry]

(II) For every, $x, y, z \in X$, if $x \leq y$ and $y \leq z$ then $x \leq z$
[transitivity]

The set $(X, \leq)$ is then called a partially ordered set, we shall take $x \leq y$ and $y \geq x$ as synonymous. For simplicity $(X, \leq)$ is written as X .

For any set X , the inclusion relation $\subseteq$ is a partial order in $\mathcal{P}(X)$.

Linear order: $\rightarrow$ If for every $x, y \in X$, $x \leq y$ or $y \leq x$ then $\leq$ is said to be ~~total~~ linear order relation in X .

For example: In the set $X = \{1, 2, 3\}$, $A = \{1, 2\} \in \mathcal{P}(X)$ and $B = \{2, 3\} \in \mathcal{P}(X)$, but neither $A \subseteq B$ nor $B \subseteq A$.

~~well order~~ In the set $\mathcal{P}(X)$, $\subseteq$ is a linear order.

part of P.1.

Well order: $\rightarrow$ Let R is a relation in X , then R is called a well-ordering which for each subset A of X has a unique R first element. Then X is called R -well-ordered.

When R is a partial order then the uniqueness is obtained from antisymmetry.

Let $(X, \leq)$ be any partially ordered set and X has an element a such that for each x in X , $a \leq x$. then a is called the first or least element of X .

Thus a is called the first element of X if and only if a is comparable with all other elements of X and no element of X is a predecessor of a .

Let $(X, \leq)$ be a partially ordered set and $A \subseteq X$. Then $l \in X$ is called a lower bound of A if $l \leq x$ is true for each element x of A .

An element $u \in X$ is called an upper bound of $A (\subseteq X)$ if $x \leq u$ is true for each member x of A .

Let $(X, \leq)$ be any partially ordered set and X has an element a such that for each x in X , $a \leq x$ then a is called the first or least element of X .

Thus a is called the first element of X if and only if a is comparable with all other elements of X and no element of X is a predecessor of a .

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Differential Calculus (partial differentiation)

Q.1. If $u = \cos^{-1} \frac{x-y}{x+y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$

Soln: We have $\cos u = \frac{x-y}{x+y}$

It is clear that $\frac{x-y}{x+y}$ is of zero degree and hence $\cos u$ is a homogeneous function of ~~degree~~ zero degree.

$\therefore$ By Euler's theorem,

$$x \frac{\partial (\cos u)}{\partial x} + y \frac{\partial (\cos u)}{\partial y} = n \cos u = 0 \times \cos u = 0$$

$$\Rightarrow -x \sin u \cdot \frac{\partial u}{\partial x} - y \sin u \cdot \frac{\partial u}{\partial y} = 0 \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \quad \text{Proved}$$

Q.2. If $u = \sin^{-1} \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

Soln: We have $\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

Since, $\sin u$ is a homogeneous function of zero degree.

$\therefore$ By Euler's theorem,

$$x \frac{\partial (\sin u)}{\partial x} + y \frac{\partial (\sin u)}{\partial y} = 0 \times \sin u$$

$$\Rightarrow \cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 0$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

Proved-

Q.3. If $u = \cos^{-1} \frac{x+y}{\sqrt{x+y}}$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$$

Soln: We have, $u = \cos^{-1} \frac{x+y}{\sqrt{x+y}}$

$$\Rightarrow \cos u = \frac{x+y}{\sqrt{x+y}}$$

$$\Rightarrow \cos u = \frac{x(1+\frac{y}{x})}{\sqrt{x}(1+\sqrt{\frac{y}{x}})}$$

$$\Rightarrow \cos u = \frac{x^{\frac{1}{2}}(1+\frac{y}{x})}{1+\sqrt{\frac{y}{x}}} = V \text{ (say)} \quad \text{--- (I)}$$

We find that V is a homogeneous function of two independent variable x and y and of degree $\frac{1}{2}$.
Hence applying Euler's theorem for the function V
We have

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = nV = \frac{1}{2}V \quad \text{--- (II)}$$

From I, $V = \cos u$

$$\therefore \frac{\partial V}{\partial x} = -\sin u \frac{\partial u}{\partial x} \text{ and } \frac{\partial V}{\partial y} = -\sin u \frac{\partial u}{\partial y}$$

Substituting these values in II, we get

$$-x \sin u \frac{\partial u}{\partial x} - y \sin u \frac{\partial u}{\partial y} = \frac{1}{2} \cos u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \cot u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$$

Proved

Differential Calculus (partial differentiation)

Q.4. If $u = \sin\left(\frac{x+y}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

Soln. $\sin u = \frac{x+y}{x+y} = v$ (say)

Since v is a homogeneous function of x and y of degree $\frac{1}{2}$.

$\therefore$ By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$\Rightarrow x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$\Rightarrow x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \quad \text{proved.}$$

Q.5. If $u = \sin(\sqrt{x+y})$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(\sqrt{x+y})\cos(\sqrt{x+y})$$

Soln. We have, $u = \sin(\sqrt{x+y}) \Rightarrow \sin^2 u = \sqrt{x+y}$

$$\Rightarrow \sin^2 u = \sqrt{x+y} = V \text{ (say)} \quad \text{--- (I)}$$

We find that V is a homogeneous function of two independent variables x and y of degree $\frac{1}{2}$.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = nV = \frac{1}{2}V \quad \text{--- (II)}$$

From (I), $V = \sin^2 u$

$$\therefore \frac{\partial V}{\partial x} = \frac{1}{\sqrt{1-u^2}} \frac{\partial u}{\partial x} \text{ and } \frac{\partial V}{\partial y} = \frac{1}{\sqrt{1-u^2}} \frac{\partial u}{\partial y}$$

Substituting these two values in II, we get

$$\frac{x}{\sqrt{1-u^2}} \frac{\partial u}{\partial x} + \frac{y}{\sqrt{1-u^2}} \frac{\partial u}{\partial y} = \frac{1}{2}\sin^2 u$$

$$\Rightarrow \frac{1}{\sqrt{1-u^2}} \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) = \frac{1}{2}(\sqrt{x+y})$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} (5x+5y) \sqrt{1-y^2}$$

$$= \frac{1}{2} (5x+5y) \sqrt{1-\sin^2(5x+5y)}$$

Hence, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} (5x+5y) \cos(5x+5y)$ proved

Q.6. If $u = \sin \frac{x^2+y^2}{x+y}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$

Soln: we have, $u = \sin \frac{x^2+y^2}{x+y}$

$$\Rightarrow \sin u = \frac{x^2+y^2}{x+y} = \frac{x^2(1+\frac{y^2}{x^2})}{x(1+\frac{y}{x})} = x \phi\left(\frac{y}{x}\right) = V, \text{ say}$$

Now, V is a homogeneous function of degree 1

$\therefore$ By Euler's theorem, we get

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = 1 \cdot V$$

But $V = \sin u$

$$\therefore \frac{\partial V}{\partial x} = \cos u \frac{\partial u}{\partial x} \text{ and } \frac{\partial V}{\partial y} = \cos u \frac{\partial u}{\partial y}$$

Substituting these values in (1), we get

$$\therefore x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$$
 proved

Q.7. If $\sin u = \frac{x^3+y^3}{x-y}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u$

Soln: Since, $\sin u = \frac{x^3+y^3}{x-y}$

Here, we see that $\sin u$ is a homogeneous function of the second degree.

$\therefore$ By Euler's theorem, we get

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = 2 \sin u$$

$$\Rightarrow x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 2 \sin u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u$$

proved